Solution 3

1. A function on $[a, b]$ is called Hölder continuous at $x \in [a, b]$ if there are $\alpha \in (0, 1), L$ and δ such that $|f(y) - f(x)| \leq L|y - x|^\alpha$ for all $y \in [a, b], |y - x| < \delta$. Prove that Theorem 1.5 holds when "Lipschitz continuous" is replaced by "Hölder continuous".

   Solution Just like the Lipschitz case, the only difference is the way to treat the term

$$II = \int \chi_A D_n(z)(f(x+z) - f(x))\, dz \ .$$

   By the Holder condition, we have, for $z, |z| \leq \delta,\ \delta \leq \min\{\delta_0, \delta_1\}$,

$$\begin{aligned} |II| &\leq \int_{-\delta}^{\delta} \frac{|\sin(n+1/2)z|}{|\sin z/2|} |f(x+z) - f(x)|\, dz \\ &\leq \frac{L}{2\pi} \int_{-\delta}^{\delta} |z|^{\alpha-1}\, dz \\ &= \frac{4\delta^\alpha L}{\alpha\pi} \ . \end{aligned}$$

   Now, it suffices to choose δ such that

$$\frac{4\delta^\alpha L}{\alpha\pi} < \frac{\varepsilon}{2} \ , \ \delta \leq \min\{\delta_0, \delta_1\} \ ,$$

   to finish the job.

2. Let f be a function defined on (a, b) and $x_0 \in (a, b)$.

   (a) Show that f is Lipschitz continuous at x_0 if its left and right derivatives exist at x_0.

   (b) Construct a function Lipschitz continuous at x_0 whose one sided derivatives do not exist.

   Solution. (a) Let $\alpha = f'_+(x_0)$. For $\varepsilon = 1 > 0$, there exists δ_1 such that

$$\left| \frac{f(x+z) - f(x)}{z} - \alpha \right| < 1 \ ,$$

   for $0 < z < \delta_1$. It follows that

$$|f(x+z) - f(x)| \leq |f(x+z) - f(x) - \alpha z| + |\alpha z| \leq (1 + |\alpha|)|z| \ .$$

   Similarly,

$$|f(x+z) - f(x)| \leq (1 + |\alpha|)|z| \ , \quad z \in (-\delta_2, 0) \ .$$

   We conclude that $|f(x+z) - f(x)| \leq (1 + \delta)|z|, \quad z \in (-\delta, \delta),\ \delta = \min\{\delta_1, \delta_2\}$.

   (b) The function $f(x) = x \sin \frac{1}{x}$ $(x \neq 0)$ and $= 0$ at $x = 0$. It is Lipschitz continuous at $x_0 = 0$ with $L = 1$ but both one-sided derivatives do not exist.

3. Let f be a function defined on $(a, b]$ which is integrable on $[c, b]$ for all $c \in (a, b)$. It is called improperly integrable over $(a, b]$ if

$$\lim_{c \to a^+} \int_c^b |f|$$

exists. When this happens,

$$\lim_{c\to a^+} \int_c^b f$$

also exists and we define the improper integral of f over $(a, b]$ to be

$$\int_a^b f = \lim_{c\to a^+} \int_c^b f \ .$$

(a) Show that if f is integrable on $[a, b]$, its improper integral also exists and is equal to it usual integral.

(b) Show that Riemann-Lebesgue Lemma holds for improperly integrable functions.

Solution. (a) Using

$$\int_a^c f \le (c-a)M, \quad M = \sup |f| \ ,$$

for $\varepsilon > 0$,

$$\left| \int_c^b |f| - \int_a^b |f| \right| = \int_a^c |f| \le (c-a)M < \varepsilon,$$

for all c, $|c - a| < \varepsilon/M$. Therefore,

$$\lim_{c\to a^+} \int_c^b |f| = \int_a^b |f| \ .$$

(b) For $\varepsilon > 0$, fix $c \in (a, b)$ such that

$$\int_a^c |f| < \varepsilon/2 \ .$$

The function f is integrable on $[c, b]$ and Riemann-Lebesgue Lemma applies to get

$$\left| \int_c^b f(x)e^{inx}dx \right| < \frac{\varepsilon}{2}, \quad n \ge n_0,$$

for some n_0. It follows that

$$\left| \int_a^b f(x)e^{inx}dx \right| \le \int_a^c |f| + \left| \int_c^b f(x)e^{inx}dx \right| < \varepsilon \ , \quad \forall n \ge n_0 \ .$$

4. Optional. Show that

$$-\log \left| 2\sin\frac{x}{2} \right| \sim \cos x + \frac{\cos 2x}{2} + \frac{\cos 3x}{3} + \cdots \ .$$

Suggestion. I leave it to you to verify this function is 2π-periodic and improperly integrable. The calculation of a_0 is tricky, involving the definite integral $I = \int_0^{\pi/2} \log \sin t dt$. To evaluate it use $\sin t = 2 \sin t/2 \cos t/2$ and eventually show $I = -\frac{\pi}{2} \log 2$.

Solution. I leave out the verification of periodicity and improper integrability. This is an even function, so its Fourier series is a cosine series. Let

$$\pi a_0 = \int_0^\pi \log\left(2\sin\frac{x}{2}\right) dx = \pi \log 2 + Y, \quad Y = \int_0^\pi \log \sin\frac{x}{2} dx.$$

We have

$$\begin{aligned} Y &= 2\int_0^{\pi/2} \log \sin t dt \\ &= 2\int_0^{\pi/2} \log\left(2\sin\frac{t}{2}\cos\frac{t}{2}\right)dt \\ &= \pi\log 2 + 2\int_0^{\pi/2}\log\sin\frac{t}{2}dt + 2\int_0^{\pi/2}\log\cos\frac{t}{2}dt \\ &= \pi\log 2 + 2\int_0^{\pi/2}\log\sin\frac{t}{2}dt + 2\int_{\pi/2}^{\pi}\log\sin\frac{t}{2}dt \\ &= \pi\log 2 + 2Y, \end{aligned}$$

so $Y = -\pi\log 2$. It follows that $a_0 = 0$. The calculations for a_n make use of by parts to get

$$a_n = \frac{1}{n\pi}\int_0^{\pi}\frac{\sin nx\cos(x/2)}{\sin(x/2)}dx$$

first, then by

$$\sin nx\cos\frac{x}{2} = \frac{1}{2}\left(\sin(n+\frac{1}{2})x + \sin(n-\frac{1}{2})x\right)$$

and finally use Property 3 of the Dirichlet kernel.

5. Let a_n, b_n be the Fourier coefficients of some $f \in R_{2\pi}$.

   (a) Show that for each $r \in [0,1)$, the trigonometric series given by

   $$a_0 + \sum_{k=1}^{\infty} r^n(a_n\cos nx + b_n\sin nx)$$

   is uniformly convergent to some function in $C_{2\pi}$. Denote this function by $f_r(x)$.

   (b) Show that

   $$f_r(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}P_r(z)f(x+z)dz,$$

   where the **Poisson kernel** P_r is given by

   $$P_r(z) = \frac{1-r^2}{1-2r\cos z + r^2}\ .$$

   (c) Let f be continuous at x. Show that $\lim_{r\to 1} f_r(x) = f(x)$.

The treatment is parallel to that for the Dirichlet kernel (the parameter n is now replaced by r), but differs at the final step; we do not need Lipschitz continuity. Think about it.

Solution. Look up [SS]. We don't need Lipschitz continuity because Poisson kernel is positive, so the analog of Property IV of the Dirichlet kernel does not hold, which is good news.

6. (a) Can you find a cosine series which converges uniformly to the sine function on $[0,\pi]$? If yes, find one.

   (b) Is the series in (a) unique?

(c) Can you find a cosine series which converges pointwisely to the sine function on $[-a, \pi]$ where a is a number in $(0, \pi)$?

Solution. (a) Yes, extend the sine function on $[0, \pi]$ to $|\sin x|$, an even, 2π-periodic function. Since it is continuous, piecewise C^1, its cosine series converges uniformly to this extended function. In particular, this cosine series converges uniformly to $\sin x$ on $[0, \pi]$. (b) Yes, there is only one way to extend $\sin x$ as an even function. (c) No, can't have even extension. (When a function is the pointwise limit of an even function, it must be even.)

7. Let f be an integrable function on $[-\pi, \pi]$. Show that for each $\varepsilon > 0$, there exists a trigonometric polynomial p satisfying $p < f$ on $[-\pi, \pi]$ and

$$\int_{-\pi}^{\pi} |f - p| < \varepsilon \ .$$

Solution. Given $\varepsilon > 0$, we can find a continuous function $g + \varepsilon_1 < f$ satisfying

$$\int_a^b |f - g| < \frac{\varepsilon}{4}$$

for a small $\varepsilon_1 > 0$. (g comes from modifying a step function constructed using a Darbourx lower sum.) Then we find a trigonometric polynomial q satisfying $|g(x) - q(x)| < \min\{\varepsilon_1/2, \varepsilon/4(b-a)\}$, so

$$\int_a^b |g - q| < \frac{\varepsilon}{4}.$$

The function $p = q + \min\{\varepsilon_1/2, \varepsilon/(4(b-a)\}$ satisfies our requirement.

Note. Weierstrass Theorem asserts every continuous function can be approximated by polynomials in $[a, b]$. Here it is shown that every integrable function can be approximated by polynomials in integral sense (that is, in average sense).

8. Optional. Show that there is a countable subset of $C[a, b]$ such that for each $f \in C[a, b]$, there is some $\varepsilon > 0$ such that $\|f - g\|_\infty < \varepsilon$ for some g in this set. Suggestion: Take this set to be the collection of all polynomials whose coefficients are rational numbers.

Solution. Let P_N the collection of all polynomials of the form $p(x) = a_0 + a_1 x + \cdots + a_N x^n$ where $a_j, j = 0, \cdots, N$ are rational numbers. The map $p \mapsto (a_0, a_1, \cdots, a_N)$ is a one-to-one correspondence between P_N and $\mathbb{Q}^N$, which is countable. As the countable union of countable sets is again countable, $P = \cup_{N=0}^{\infty} P_N$ is also countable. Now, by Weierstrass theorem, for each $f \in [a, b]$ and $\varepsilon > 0$, there exists a polynomial q (with real coefficients) such that $\|f - q\|_\infty < \varepsilon/2$. We may approximate q by a polynomial p from P such that $\|q - p\|_\infty$. It follows that $\|f - p\|_\infty \le \|f - q\|_\infty + \|q - p\|_\infty < \varepsilon$.

9. Let f be continuous on $[a, b] \times [c, d]$. Show that for each $\varepsilon > 0$, there exists a polynomial $p = p(x, y)$ so that

$$\left\|f - p\right\|_\infty < \varepsilon, \quad \text{in } [a, b] \times [c, d].$$

In fact, this result holds in arbitrary dimension.

Solution. Just like we approximate a continuous function by continuous piecewise linear function in the one dimensional case, WLOG we may assume $f(x, y)$ is doubly 2π-periodic, uniformly Lipschitz continuous in $[-\pi, \pi]^2$. For every $\varepsilon > 0$, we can find an N such that for all y

$$|f(x, y) - \sum_{k=1}^{N} (a_k(y) \cos kx + b_k(y) \sin kx| < \varepsilon, \quad \forall x \ . \tag{1}$$

(See the last paragraph.) From the expression

$$a_k(y) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x,y)\cos kx dx$$

we see that each a_k is in $C_{2\pi}$. By Weierstrass Approximation Theorem, we can find polynomials $p_k(x)$ such that $|a_k(x) - p_k(x)| < \varepsilon/N$ for all x, and q_k for b_k similarly. It follows that

$$\begin{aligned}
& |f(x,y) - \sum_{k=1}^{N}(p_k(x)\cos kx + q_k(x)\sin kx)| \\
\le\ & \left| f - \sum_{k=1}^{N}(a_k(x)\cos kx + b_k(x)\sin kx)\right| \\
& + \left|\sum_{k=1}^{N}(a_k(x)\cos kx + b_k(x)\sin kx) - \sum_{k=1}^{N}(p_k(x)\cos kx + q_k(x)\sin kx)\right| \\
<\ & \varepsilon + 2N \times \frac{\varepsilon}{N} \\
=\ & 3\varepsilon.
\end{aligned}$$

Finally, we approximate $\cos kx$ and $\sin kx$ by polynomials in x to complete the job.

We justify (1) by a compactness argument. For each fixed y, the function $x \mapsto f(x,y)$ is uniformly Lipschitz continuous. For $\varepsilon > 0$, there exists some natural number N_y depending on y such that

$$|f(x,y) - \sum_{k=1}^{N_y}(a_k(y)\cos kx + b_k(y)\sin kx| < \varepsilon, \quad \forall x\ .$$

By continuity, there is some interval $(y - \delta_y, y + \delta_y)$ so that

$$|f(x,z) - \sum_{k=1}^{N_y}(a_k(z)\cos kx + b_k(z)\sin kx| < \varepsilon, \quad \forall x\ ,\ \text{and } z \in (y - \delta_y, y + \delta_y)\ .$$

All these intervals $\{(y - \delta_y, y + \delta_y)\}, y \in [-\pi, \pi]$, form an open covering of $[-\pi, \pi]$. By Open Covering Theorem (I taught it in MATH2050 many years ago. Supposedly it is still covered in this course), we can find a finite subcover $\{(y_k - \delta_{y_k}, y_k + \delta_{y_k})\},\ k = 1, \cdots, M$. Then (1) holds by taking $N = \max\{N_{y_1}, \cdots, N_{y_M}\}$.